PARABOLIC EQUATIONS OF THE SECOND ORDER(1)

BY

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Introduction. In this paper we develop several aspects of the theory of second order parabolic equations of the form

$$(0.1) \qquad Lu \equiv \sum_{i=1}^{n} a_i(x, t) \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} = f$$

from a unified point of view, namely, the extensive use of Green's function. Our main interest is concerned with the first mixed boundary problem (for definition, see §4) for f linear or nonlinear in u, Harnack theorems, etc.

Most of our methods are known. Green's function for the heat equation in rectangular domains was constructed already by E. E. Levi [21] (in particular §7) in 1907 and was used by him to derive some existence theorems. An extensive use of Green's function for more general parabolic equations was made by Gevrey in his fundamental paper [16] (in particular §§4, 4*, 24, 28, 39, 40, 41). The analogue of the Harnack convergence theorem was first proved for the heat equation in Levi's paper [21, pp. 386–387]. Nonlinear existence problems were considered by Gevrey [16, §28] by reducing them with the aid of Green's function to integral equations and then applying successive approximations. A more detailed survey on the older literature concerned with Green's function is to be found in the book of Ascoli-Burgatti-Giraud [1].

In this paper we use all the above mentioned methods and a few new ones to treat more general problems than those considered in earlier papers. Essential use of Dressel's fundamental solutions for general linear second order parabolic equations [10; 11] enables us to perform this extension. We give below a brief description of our results and their connection to previous papers.

In §1 we construct Green's function for linear second order parabolic equations with smooth coefficients in an $(n+1)$-dimensional rectangle. In this construction we employ the fundamental solutions constructed by Dressel [11]. Green's function for the heat equation in one dimension was constructed by Levi [21] by reflecting the fundamental solution $t^{-1/2}\exp(-x^2/4t)$ with respect to the x-variable; our method is an extension of Levi's method.

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In §2 we prove some properties of Green's function $G(x, t; \xi, t)$constructed in §1. We prove a symmetry relation between Green's function of two equations, one of which is the adjoint of the other. Differentiability properties of G with respect to all the arguments are discussed, and the behavior of G near the boundary is studied. Finally, the first mixed boundary problem for the equation (0.1) with $f \equiv 0$ in rectangular domains is solved.

In §3 we briefly discuss the Harnack inequality for non-negative solutions of (0.1) with $f \equiv 0$ (the special case of the heat equation was proved by Hadamard [17]), and the Harnack theorem concerning uniform convergence of solutions.

In §4 we define notions analogous to "super-harmonic" and "barriers" and prove various properties. We then construct, following Poincaré's *méthode de balayage*, a generalized solution of the first mixed boundary problem. Under some mild assumptions on the boundary, we construct barriers and thus prove that the generalized solutions are genuine solutions. The special case of the heat equation in one dimension was considered by Pini [28]; however, Pini assumed that the first mixed boundary value problem for the heat equation in domains with smooth boundary can be solved (a result which was established, for instance, by Gevrey [16]), while we do not make such an assumption; our treatment is self-contained. The decisive step which enables us to apply the Poincaré's *méthode de balayage* is the construction of a solution of the first mixed boundary problem in rectangular domains and for *discontinuous* boundary values. The idea of applying Perron's method (which slightly differs from that of Poincaré) to the heat equation was used already by W. Sternberg [34].

In §5 we discuss the question of existence of solution of the first mixed boundary problem for equations of the form $Lu = f(x, t, u)$ in cylindrical domains. Our results contain as very special cases those of [3; 23; 33]. A method which uses the Schauder type estimates and the Schauder-Leray method in the theory of elliptic equations was used by Ciliberto [7; 8; 9] for the equation $Lu = f(x, t, u)$ $(n = 1)$ in the case $f = f(x, t, u, \partial u/\partial x)$. The author considered in [15] the noncylindrical case of n dimensions and $f = f(x, t, u, \text{grad } u)$. However, the results and methods of §5 are not contained in all the above mentioned works.

The results of this paper are all based on the study of Green's function. However there are other approaches in the study of second order parabolic equations. We first mention the finite difference method which leads both to existence theorems and numerical methods to calculate solutions; see [18; 20; 26; 31; 32; 35; 36]. As a second approach we mention the method of a priori estimates. This method is analogous to Schauder's method for elliptic equations. Both the analogous estimates and the existence theorem for the first mixed boundary problem were established by Richard B. Barrar [2] in his unpublished thesis for *general* second order parabolic equations. Later on

these estimates were reproved by Ciliberto [6] in the case of one dimension. More recently the author [14] has derived these estimates in a simpler way and also derived existence theorems which contain both that of Barrar and Theorem 3 of this paper. We also remark that both the Harnack inequality and the Harnack uniform convergence theorem follow immediately (for general second order parabolic equations with Hölder continuous coefficients) from the Schauder type estimates of [2] or [14] with the aid of the maximum principle [24].

DEFINITION. A function g is said to satisfy a Hölder condition with exponent α $(0<\alpha<1)$ and with coefficient H in a set S if for every pair of points P, Q and S

$$|g(P)-g(Q)| \leqq H|P-Q|^{\alpha}. \tag{0.2}$$

g is then also said to be Hölder continuous (exponent α) in S. g is Hölder continuous at a point P if (0.2) holds for all Q in a neighborhood of P.

1. **Construction of Green's functions for rectangular domains.** Let R be an n-dimensional cube in the space of real variables $x=(x_1, \cdots, x_n)$ and let D be the topological product of R with a real interval $0<t<T$. Consider the equation

$$Lu \equiv \sum_{i=1}^{n} a_i(x, t)\frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} b_i(x, t)\frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} = 0. \tag{1.1}$$

We shall make use of the following assumptions:

(A) L is defined and uniformly parabolic in D, that is, there exists a positive constant g such that for every $(x, t)\in D$ and for all real vectors $\xi=(\xi_1, \cdots, \xi_n)$

$$\sum_i a_i(x, t)\xi_i^2 \geqq g\sum_i \xi_i^2. \tag{1.2}$$

(B) The functions

$$a_i,\ \frac{\partial}{\partial t}a_i,\ \frac{\partial}{\partial x_k}a_i,\ \frac{\partial^2}{\partial x_k \partial x_m}a_i,\ b_i,\ \frac{\partial}{\partial x_k}b_i,\ c \tag{1.3}$$

are Hölder continuous in $\overline{D}$ (the closure of D). Throughout this paper we shall denote by M a bound on the functions in (1.3).

REMARK 1. All the results of this paper hold also for the class of parabolic equations which can locally be reduced by a one-to-one transformation defined on D to equations of the form (1.1). This class includes general second order parabolic equations with space-dimension $n=2$, the transformation being

$$x_2' = x_2, \qquad x_1' = -\int^{x_2} \frac{a_{12}(x_1, s)}{a_{11}(x_1, s)}\,ds + Ax_1$$

where A is sufficiently large and a_{ij} is the coefficient of $\partial^2/\partial x_i \partial x_j$.

REMARK 2. The assumption that $\partial b_i/\partial x_k$ exist and are Hölder continuous in $\overline{D}$ is not used in the construction of Green's function in the present section. However, we make an extensive use of that assumption in proving Lemmas 3–7 (§2). For simplicity, we make that assumption also in the present section.

Under the assumptions (A), (B) we shall now construct Green's function in the domain D. For simplicity we may assume that R is the cube $-l<x_i<l$ $(i=1, \cdots, n)$. We first extend the definition of the coefficients of L to the whole strip $0 \leqq t \leqq T$ in the following way:

Let $-l \leqq x_i \leqq l$ $(2 \leqq i \leqq n)$, $0 \leqq t \leqq T$: If $-l<x_1<l$, $s(2l-x_1, x_2, \cdots, x_n, t) = s(x_1, x_2, \cdots, x_n, t)$ for $s=a_i$ $(i=1, 2, \cdots, n)$, $s=b_i$ $(i=2, \cdots, n)$, $s=c$, and $b_1(2l-x_1, x_2, \cdots, x_n, t) = -b_1(x_1, x_2, \cdots, x_n, t)$. Next, if $-l \leqq x_1 < 3l$, $s(x_1+4kl, x_2, \cdots, x_n, t) = s(x_1, x_2, \cdots, x_n, t)$ $(k=\pm 1, \pm 2, \cdots)$ for $s=a_i, b_i, c$. Having defined the coefficients of L for $-\infty<x_1<\infty$, $-l \leqq x_i \leqq l$ $(2 \leqq i \leqq n)$, $0 \leqq t \leqq T$, we proceed in the same way to extend the coefficients of L along the x_2-axis. Continuing in this way, we finally obtain L extended to the whole strip $0 \leqq t \leqq T$.

Let B be the space $-\infty<x_i<\infty$ $(i=1, \cdots, n)$ and let S be the set of points x which are of the form $(x_1, \cdots, x_{i-1}, (2k+1)l, x_{i+1}, \cdots, x_n)$ for some integers i and k. The coefficients of L (extended in the above manner) have the following properties:

(α) L is uniformly parabolic in the strip $0 \leqq t \leqq T$, that is, (1.2) holds for all (x, t) in the strip.

(β) The functions

$$a_i, \quad \frac{\partial}{\partial t} a_i, \quad \frac{\partial^2}{\partial x_k^2} a_i, \quad c$$

are Hölder continuous in the whole strip; the functions

$$\frac{\partial}{\partial x_k} a_i, \quad b_i$$

are bounded and piecewise continuous, the discontinuity is of the first kind and may happen only on the topological product of S with $0 \leqq t \leqq T$. All the above functions are Hölder continuous at each point (x, t), $x \in B-S$, $0<t<T$.

The conditions (α) (β) are sufficient to ensure the existence of a fundamental solution for L in the strip $0 \leqq t \leqq T$, with a certain discontinuity at the points (x, t) with $x \in S$. Indeed, from Dressel's papers [10; 11], in which the conditions (α) (β), with the set S replaced by the empty set, are assumed, one gets the following: Denote by (A_i) the matrix inverse to (a_i), and let $\sigma(x, t; \xi) = \sum A_i(x, t)\xi_i^2$. We have $\sigma(x, t; \xi) \geqq g'|\xi|^2$ where g' depends on g and M. Denote

$$
(1.4)\qquad
\begin{aligned}
U(x, t; \xi, \tau) &= \begin{cases} (t-\tau)^{-n/2} \exp[-\sigma(x, t; x-\xi)/4(t-\tau)] & \text{if } t > \tau, \\ 0 & \text{if } t < \tau, \end{cases} \\
F(x, t) &= A \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left(\frac{\sigma(x, t; x-\xi)}{|x-\xi|^2} \right)^{-n/2} \\
&\qquad \sin^{n-2}\theta_1, \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} d\theta_1 d\vartheta_2 \cdots d\vartheta_{n-2}
\end{aligned}
$$

where $A = 2^n \int_0^\infty w^{n-1} \exp(-w^2) dw$. The function

$$
(1.5)\qquad Z(x, t; \xi, \tau) = \frac{U(x, t; \xi, \tau)}{F(\xi, \tau)}
$$

is the "essential" part of the fundamental solution $\Gamma(x, t; \xi, \tau)$, and

$$
(1.6)\qquad \Gamma(x, t; \xi, \tau) = Z(x, t; \xi, \tau) + \int_\tau^t \int_B Z(x, t; s, r) f(s, r; \xi, \tau) ds dr,
$$

where f is the solution of the integral equation

$$
(1.7)\qquad f(x, t; \xi, \tau) = L(Z(x, t; \xi, \tau)) + \int_\tau^t \int_B L(Z(s, t; s, r)) f(s, r; \xi, \tau) ds dr.
$$

Noting that

$$
(1.8)\qquad |L[Z(x, t; s, r)]| \leqq \frac{\text{const.}}{(t-r)^{(n+1)/2}} \exp\left\{ -\text{const.} \frac{|x-s|^2}{4(t-r)} \right\}
$$

and using Lemma 2 in §2, we get

$$
(1.9)\qquad |f(x, t; \xi, \tau)| \leqq \frac{H_0}{(t-\tau)^{(n+1)/2}} \exp\left\{ -\frac{H|x-\xi|^2}{4(t-\tau)} \right\},
$$

where H_0, H depend only on g and M.

In the following lemma we mention further properties of Γ, valid under the assumptions (α), (β), which will be used later. The proof will be omitted.

LEMMA 1. *Γ is continuous and $\partial\Gamma/\partial x_i$, $\partial^2\Gamma/\partial x_i \partial x_j$, $\partial\Gamma/\partial t$ are piecewise continuous for $(x, t)(\neq(\xi, \tau))$ in the strip $0 \leqq t \leqq T$; the discontinuity is of the first kind and may occur only if $x \in S$, $0 \leqq t \leqq T$.*

For simplicity we shall first construct Green's function in the case $n=1$. By Lemma 1 it follows that the functions

$$
(1.10)\qquad v_k(x, t; \xi, \tau) = \Gamma(x + 4kl, t; \xi, \tau), \quad w_k(x, t; \xi, \tau) = \Gamma(2l - x + 4kl, t; \xi, \tau)
$$

are continuous in the rectangle D together with their first derivatives with respect to x and t, and their second derivative with respect to x. Moreover, these derivatives can be extended, by continuity, to $x = \pm l$ and to $t = 0, T$.

It also follows that $v_k(l, t; \xi, \tau) = w_k(l, t; \xi, \tau)$ and $v_k(-l, t; \xi, \tau) = w_{k-1}(-l, t; \xi, \tau)$.

From the way we extended the definition of the coefficients of L, it is clear that, as functions of x and t, v_k are solutions of (1.1). Consider the function

$$G(x, t; \xi, \tau) = \sum_{-\infty}^{\infty} v_k(x, t; \xi, \tau) - \sum_{-\infty}^{\infty} w_k(x, t; \xi, \tau). \tag{1.11}$$

For $(\xi, \tau) \in D$, each term v_k (or w_k), $k \neq 0$, together with its second x-derivatives and first t-derivative is bounded by

$$\frac{H''}{(t-\tau)^a} \exp\left\{-\frac{H' \mid x - \xi + 4kl \mid^2}{t - \tau}\right\}$$

where H', H'' and a are appropriate constants. We thus conclude that the series in (1.11) converges uniformly in D, together with its first and second x-derivatives and first t-derivative. It represents the desired Green's function, since it is a fundamental solution of the equation (1.1) in D, and since it vanishes on the lateral boundary of D, $x = \pm l$.

For arbitrary n, we can construct Green's function in a similar manner. We only write its explicit form for $n = 2$:

$$\begin{aligned} G(x, t; \xi, \tau) = \sum_{i=-\infty}^{\infty} \Bigg[\sum_{k=-\infty}^{\infty} & \Gamma(x_1^i, x_2^k, t; \xi_1, \xi_2, \tau) \\ & - \sum_{k=-\infty}^{\infty} \Gamma(x_1^i, \bar{x}_2^k, t; \xi_1, \xi_2, \tau) \Bigg] \\ - \sum_{i=-\infty}^{\infty} \Bigg[\sum_{k=-\infty}^{\infty} & \Gamma(\bar{x}_1^i, x_2^k, t; \xi_1, \xi_2, \tau) \\ & - \sum_{k=-\infty}^{\infty} \Gamma(\bar{x}_1^i, \bar{x}_2^k, t; \xi_1, \xi_2, \tau) \Bigg], \end{aligned} \tag{1.12}$$

where $x_\lambda^k = x_\lambda + 4kl$, $\bar{x}_\lambda^k = 2l - x_\lambda + 4kl$ $(\lambda = 1, 2)$.

REMARK. In case R is an n-dimensional rectangle, we can still construct Green's function in the above manner, by first using an affine transformation to transform R into an n-dimensional cube.

DEFINITION. By an $(n+1)$-*dimensional rectangle* we shall always mean a rectangle all of whose edges are parallel to the coordinate axes.

2. **Properties of Green's functions for rectangular domains.** In this section R will denote the n-dimensional rectangle $-l_i < x_i < l_i$ $(i = 1, \cdots, n)$. We shall prove some properties of Green's function $G(x, t; \xi, \tau)$ for the $(n+1)$-dimensional domain D, constructed in the preceding section. In the sequel, we shall make use of the following lemma [11].

LEMMA 2. *If* $0<\alpha, \beta<n/2+1$, $k>0$, *then*

$$\int_\tau^t \int_R \frac{\exp(-k\,|\,x-\xi\,|^2/4(s-\tau))}{(s-\tau)^\alpha} \frac{\exp(-k\,|\,\xi-y\,|^2/4(t-s))}{(t-s)^\beta} d\xi ds \leqq A \frac{\exp(-k\,|\,x-y\,|^2/4(t-\tau))}{(t-\tau)^{\alpha+\beta-1-n/2}},$$

where A *is a constant depending on* α, β, n *and* k.

We shall now prove:

LEMMA 3. *The integrals*

$$\begin{aligned}
&\int_R \left| \frac{\partial^2}{\partial\xi_i\partial\xi_j} G(\xi, \tau; x, 0) \right| dx, \\
&\int_{R_m}\int_0^\tau \left| \left[\frac{\partial^2}{\partial\xi_i\partial\xi_j} \frac{\partial}{\partial x_k} G(\xi, \tau; x, t) \right]_{x_m=\pm l_m} \right| dt \frac{dx}{dx_m}, \\
&\int_R \left| \frac{\partial}{\partial\tau} G(\xi, \tau; x, 0) \right| dx, \\
&\int_{R_m}\int_0^\tau \left| \left[\frac{\partial}{\partial\tau} \frac{\partial}{\partial x_k} G(\xi, \tau; x, t) \right]_{x_m=\pm l_m} \right| dt \frac{dx}{dx_m}
\end{aligned} \tag{2.1}$$

(*where* $dx/dx_m = dx_1 \cdots dx_{m-1}dx_{m+1} \cdots dx_n$) *exist and are continuous functions of* (ξ, τ), $(\xi, \tau)\in D$. *Here,* R_m *is the* $(n-1)$*-dimensional rectangle* $-l_i \leqq x_i \leqq l_i$, $i=1, \cdots, m-1, m+1, \cdots, n$.

We shall sketch the proof without going into all the details. Let us first consider the integrands in (2.1) with G replaced by Γ. From (1.6) and the definition of Z in (1.5) it is clear that it is enough to consider the functions (2.1) with G replaced by

$$\int_t^\tau \int_B Z(\xi, \tau; s, r) f(s, r; x, t) ds dr. \tag{2.2}$$

Furthermore, developing f into a Neumann's series (from (1.7)), one can show (by a technique similar to [10; 11]) that we may still replace the f in (2.2) by LZ. We thus have to consider the functions (2.1) with G replaced by

$$\int_t^\tau \int_B Z(\xi, \tau; s, r) LZ(s, r; x, t) ds dr \equiv I. \tag{2.3}$$

Using the proofs of [10, Theorems 1,2], we conclude that the functions

$$\frac{\partial I}{\partial\tau}, \frac{\partial^2 I}{\partial\xi_i\partial\xi_j} \text{ for } t=0, \quad \text{and} \quad \frac{\partial^2 I}{\partial x_k\partial\tau}, \frac{\partial^3 I}{\partial x_k\partial\xi_i\partial\xi_j} \text{ for } x_m = \pm l_m$$

exist and are continuous, provided $(x, t) \neq (\xi, \tau)$. Hence, the integrals (2.1) with G replaced by I exist and are continuous in (ξ, τ), $(\xi, \tau) \in D$. Consequently, the same statement holds for Γ. Using the explicit formula for G, the proof of the lemma is easily completed.

REMARK 1. If we assume more regularity conditions on the coefficients of L, then we can prove more regularity properties of G. The proof of this statement (as well as the proof of Lemma 3 itself) can be given by the method of Eidelman [12, in particular, pp. 62–70].

REMARK 2. From the proof of Lemma 3 we also derive the continuity of the functions

$$\frac{\partial^2}{\partial \xi_i \partial \xi_j} \frac{\partial}{\partial x_k} G(x, t; \xi, \tau), \qquad \frac{\partial}{\partial \tau} \frac{\partial}{\partial x_k} G(x, t; \xi, \tau)$$

in the domain $(x, t) \in D$, $(\xi, \tau) \in D$ provided $(x, t) \neq (\xi, \tau)$.

We now introduce the parabolic operator L^*, the adjoint of L, defined by

$$\begin{aligned} L^*u &\equiv \sum_i \frac{\partial^2}{\partial x_i^2}(a_i u) - \sum_i \frac{\partial}{\partial x_i}(b_i u) + cu + \frac{\partial u}{\partial t} \\ &= \sum_i a_i \frac{\partial^2 u}{\partial x_i^2} + \sum_i (2a_i - b_i) \frac{\partial u}{\partial x_i} + \left(\sum_i \frac{\partial^2 a_i}{\partial x_i^2} - \sum_i \frac{\partial b_i}{\partial x_i} + c \right) \\ &\quad + \frac{\partial u}{\partial t}. \end{aligned} \tag{2.4}$$

If the coefficients of L satisfy the assumptions (A), (B), then the coefficients of L^* also satisfy these assumptions. Consequently, we can apply the procedure of §1 and construct Green's function $G^*(x, t; \xi, \tau)$ for L^* in D. G^* satisfies, as a function of (x, t), the equation $L^*G^* = 0$, it vanishes on the lateral boundary of D, and for $t > \tau$. We shall prove a symmetry relation between G and G^*.

LEMMA 4. *If* $\tau' > \tau$,

$$G(\xi', \tau'; \xi, \tau) = G^*(\xi, \tau; \xi', \tau').$$

Proof. Consider the functions

$$u(x, t) = G(x, t; \xi, \tau), \qquad v(x, t) = G^*(x, t; \xi', \tau')$$

where $\tau < \tau'$. Let σ, σ' satisfy $\tau < \sigma < \sigma' < \tau'$. Consider Green's identity

$$vLu - uL^*v = \sum_{i=1}^n \frac{\partial}{\partial x_i} B_i - \frac{\partial}{\partial t}(uv), \tag{2.5}$$

where

$$B_i = va_i \frac{\partial u}{\partial x_i} - ua_i \frac{\partial v}{\partial x_i} - uv \frac{\partial a_i}{\partial x_i} + b_i uv. \tag{2.6}$$

Since $\partial G/\partial x_i$, $\partial G^*/\partial x_i$ are continuous for $(x, t) \in \overline{D}$, $(\xi, \tau) \in \overline{D}$ provided $(x, t) \neq (\xi, \tau)$ (this follows from Lemma 1 and the explicit formulas for G, G^*), we conclude that the B_i vanish on the lateral boundary of D. Hence, integrating (2.5) with respect to x $(x \in R)$ and t $(\sigma < t < \sigma')$, we obtain

$$\int_R [G(x, \sigma'; \xi, \tau)G^*(x, \sigma'; \xi', \tau') - G(x, \sigma; \xi, \tau)G^*(x, \sigma; \xi', \tau')]dx = 0. \tag{2.7}$$

From the explicit formula for G it follows that

$$\lim_{\sigma \to \tau} [G(x, \sigma; \xi, \tau) - \Gamma(x, \sigma; \xi, \tau)] = 0 \qquad \text{if } x \neq \xi. \tag{2.8}$$

A similar statement holds for G^*. Hence, taking in (2.7) $\sigma \to \tau$ and $\sigma' \to \tau'$, and using [11, Theorem 3], we easily get

$$G(\xi', \tau'; \xi, \tau) - G^*(\xi, \tau; \xi', \tau') = 0,$$

and the proof is completed.

From Lemma 4 we can easily conclude:

LEMMA 5. *If $\xi \in \dot{R}$ (the boundary of R) and if $(x, t) \in \overline{D}$, $(x, t) \neq (\xi, \tau)$, then*

$$G(x, t; \xi, \tau) = 0, \qquad \frac{\partial}{\partial x} G(x, t; \xi, \tau) = 0.$$

If $(x, t) \neq (\xi, \tau)$ and if $(x, t) \in D$, $(\xi, \tau) \in D$, then $G(x, t; \xi, \tau)$, as a function of (ξ, τ), satisfies the equation $L^*G(x, t; \xi, \tau) = 0$. Analogous results hold for G^*.

Let $u(x)$ be a solution of (1.1) in D, and suppose that $u(x)$ is continuously differentiable in $\overline{D}$. Integrating Green's identity (2.5), with $v(x, t) = G^*(x, t; \xi, \tau)$, with respect to x $(x \in R)$ and t $(0 < t < \tau - \epsilon)$, and letting $\epsilon \to 0$, we obtain

$$\begin{aligned} u(\xi, \tau) = {} & \int_R u(x, 0)G^*(x, 0; \xi, \tau)dx + \sum_{i=1}^{n} \int_0^\tau \int_{R_i} \left[ua_i \frac{\partial G^*}{\partial x_i} \right]_{x_i = -l_i} \frac{dx}{dx_i} dt \\ & - \sum_{i=1}^{n} \int_0^\tau \int_{R_i} \left[ua_i \frac{\partial G^*}{\partial x_i} \right]_{x_i = l_i} \frac{dx}{dx_i} dt, \end{aligned} \tag{2.9}$$

where dx/dx_i and R_i are defined in Lemma 3. Using Lemmas 3 and 4 it follows that each of the integrals in (2.9) can be differentiated once with respect to τ or twice with respect to the ξ_i. Moreover, we can differentiate them under the integral sign. Hence, each of the integrals in (2.9) is a solution of (1.1). This statement is true also in case the boundary values ($u(x, 0)$, etc.) appearing in these integrals are merely bounded integrable functions.

LEMMA 6. *Consider the functions*

$$P(\xi, \tau) = \int_R p(x)G^*(x, 0; \xi, \tau)dx, \tag{2.10}$$

$$Q_{\pm i}(\xi, \tau) = \int_0^\tau \int_{R_i} \left[qa_i \frac{\partial G^*}{\partial x_i} \right]_{x_i=\pm l_i} \frac{dx}{dx_i}\, dt, \tag{2.11}$$

where $p(x)$ $(x \in R)$, $q(x, t)$ $(x \in \dot{R},\ 0<t<T)$ *are bounded integrable functions. Then* P *and* $Q_{\pm i}$ *are solutions of* (1.1), *and satisfy the following boundary conditions*:

(α) *If* $\xi \to \dot{R}$, $\tau \to t^0 > 0$, *then* $P(\xi, \tau) \to 0$. *If* $p(x)$ *is continuous at* $x = x^0 \in R$ *then* $P(\xi, \tau) \to p(x^0)$ *as* $(\xi, \tau) \to (x^0, 0)$.

(β) *If* $\xi \to \dot{R} - R_{+i}$ (*where* R_{+i} *is the face of* R *which lies on* $x_i = +l_i$), *then* $Q_{+i}(\xi, \tau) \to 0$. *If* $\tau \to 0$, $\xi \to \xi^0 \in R$, *then* $Q_{+i}(\xi, \tau) \to 0$. *If* $q(x, t)$ *is continuous at a point* $(x^0, t^0) \in R_{+i}$, *then* $Q_{+i}(\xi, \tau) \to -q(x^0, t^0)$ *as* $(\xi, \tau) \to (x^0, t^0)$.

(γ) *Statements analogous to those of* (β) *hold for* $Q_{-i}(\xi, \tau)$. (*Note that* $Q_{-i}(\xi, \tau) \to +q(x^0, t^0)$ *on* R_{-i}.)

Proof. That the functions (2.10), (2.11) satisfy (1.1) was already proved. It will be enough to prove (α) and (β). If we notice that (2.8) is valid with G, Γ replaced by G^*, Γ^*, and if we use [11, Theorem 3], then we derive the second part of (α). The first part of (α) follows from Lemma 5 and the continuity of $G^*(x, 0; \xi, \tau)$ for $(x, 0) \neq (\xi, \tau)$.

The part of (β) concerning $\xi \to \dot{R} - R_{+i}$ follows from Lemma 5. Further, if $x \in \dot{R}$, $t < \tau$, $\tau \to 0$ and $\xi \to \xi^0 \in R$, then $\partial G^*(x, t; \xi, \tau)/\partial x_j \to 0$ uniformly with respect to x. Thus, the proof of the first part of (β) is completed. The proof of the second part of (β) follows easily upon substituting

$$\tau' = \frac{(\tau - t)^{1/2}}{l_i - \xi_i}$$

and noting that as $\xi \to x^0$, $\xi_i \to l_i$.

LEMMA 7. *For every solution of* (1.1) *in* D *which is continuous in* $\overline{D}$, *the representation formula* (2.9) *holds.*

Proof. From the given values of u on the boundary, construct the right side of (2.9), and denote it by v. We wish to prove that $u \equiv v$. Now, $u - v$ vanishes on R and on the faces $R_{\pm i}$. Since it is also a solution of (1.1) we can apply the maximum principle [24, p. 175], and thus conclude that $u - v \equiv 0$.

We conclude this section with noting that the first mixed boundary problem can be solved, in terms of (2.9), not only for continuous boundary values but also for discontinuous data. If (x^0, t^0) is a point of discontinuity, and if u_+ and u_- are the supremum and infimum of the set of the limits of the boundary values when their argument (x, t) tends to (x^0, t^0), then the solution $u(x, t)$, $(x, t) \in D$, remains between u_- and u_+ as $(x, t) \to (x^0, t^0)$.

3. **Harnack's theorems.** A domain D is said to possess *A-property* if each two points in D can be connected by a continuous simple curve in D along which the t-coordinate varies monotonically.

We state the analogue of Harnack's inequality.

THEOREM 1. *Let D be an arbitrary bounded $(n+1)$-dimensional domain, and assume that L satisfies the assumptions* (A), (B) (*in* §1). *Then for any closed subdomain D^0 of D which possesses A-property there exists a positive constant B, depending only on D^0, g and M (the bound on the functions in* (1.3)), *such that for any non-negative solution of* (1.1) *in D,*

$$u(x, t) \geqq Bu(x', t') \tag{3.1}$$

for all $(x, t) \in D^0$, $(x', t') \in D^0$, provided $t \geqq t'$.

Proof. Denote by E the $(n+1)$-dimensional rectangle of the form: $-l<x_i<l$ $(i=1, \cdots, n)$ $0<t<\rho$. We may assume that $\bar{E}$ belongs to D, and we first prove the inequality (3.1) in subdomains of E. The proof of the theorem follows easily from the above special case.

For simplicity we shall consider here only the case $n=1$; the proof of the general case is quite similar. Let

$$Lu \equiv au_{xx} + bu_x + cu - u_t = 0. \tag{3.2}$$

Green's identity (2.9) takes the form

$$\begin{aligned} u(\xi, \tau) = &\int_{-l}^{l} u(x, 0)G^*(x, 0; \xi, \tau)dx + \int_0^{\tau} \left[ua \frac{\partial G^*}{\partial x} \right]_{x=-l} dt \\ &- \int_0^{\tau} \left[ua \frac{\partial G^*}{\partial x} \right]_{x=l} dt. \end{aligned} \tag{3.3}$$

A similar representation holds for $u(\xi', \tau')$, $\tau' \leqq \tau$. We wish to find a lower positive bound on $u(\xi, \tau)/u(\xi', \tau')$. Since the second and third integrands on the right side of (3.3) are non-negative and nonpositive respectively, it is permitted, in finding that bound, to replace $u(\xi, \tau)$ by

$$\int_{-l}^{l} u(x, 0)G^*(x, 0; \xi, \tau)dx + \int_0^{\tau'} \left[ua \frac{\partial G^*}{\partial x} \right]_{x=-l} dt - \int_0^{\tau'} \left[ua \frac{\partial G^*}{\partial x} \right]_{x=l} dt.$$

Consequently, it is sufficient to prove that

$$\frac{G^*(x, 0, \xi, \tau)}{G^*(x, 0; \xi', \tau')} \geqq C, \qquad (-l < x < l), \tag{3.4}$$

$$\frac{\partial G^*(\pm l, t; \xi, \tau)/\partial x}{\partial G^*(\pm l, t; \xi', \tau')/\partial x} \geqq C, \qquad (0 < t < \tau') \tag{3.5}$$

where C is a positive constant. The proof of (3.4), (3.5) can be given by ex-

tending the method of [17] and making use of the explicit formulas for G, G^*. Details are omitted.

DEFINITION. A bounded domain D is called a *normal domain* if it is bounded by hyperplanes $t=t_1$, $t=t_2$ $(t_2>t_1)$ and by a surface C between the hyperplanes, and if its boundary on $t=t_2$ is a closed domain. The *normal boundary* of D consists of C and of the boundary B which lies on $t=t_1$. We denote $\partial D=B\cup C$. Without loss of generality, we shall always take $t_1=0$, $t_2=T>0$.

In the next section we shall solve the first mixed boundary problem for normal domains. We shall then need the following theorem, which is the analogue of a classical theorem of Harnack. (In what follows we denote by $[D^0]^-$ the closure of D^0.)

THEOREM 2. *Assume that L satisfies the assumptions* (A), (B). *If $\{u_k(x, t)\}$ is a sequence of solutions of* (1.1) *which converges uniformly on the normal boundary of a normal subdomain D^0 of D $([D^0]^-\subset D)$, then it converges uniformly in D^0 to a solution $u(x, t)$ of* (1.1). *Moreover, the sequences $\{\partial u_k/\partial t\}$, $\{\partial u_k/\partial x_i\}$ and $\{\partial^2 u_k/\partial x_i\partial x_j\}$ converge uniformly in closed subsets of D^0 to $\partial u/\partial t$, $\partial u/\partial x_i$ and $\partial^2 u/\partial x_i\partial x_j$ respectively.*

Proof. Using the maximum principle [24] it easily follows that the sequence $\{u_k(x, t)\}$ converges uniformly in D^0 to a function, say, $u(x, t)$. Hence, it is sufficient to prove the rest of the theorem for subdomains which are $(n+1)$-dimensional rectangles. Using, for u_k, the representation formula (2.9), and applying Lemma 3, the proof is easily completed.

4. **The first mixed boundary problem for general domains.** In the sequel we assume that L satisfies (A), (B). We shall also assume that $c(x, t)\leqq 0$, so that the maximum principle can be applied directly. Later on we shall show that the final result is true also for general $c(x, t)$.

Let a continuous function $\psi(x, t)$ be given on the normal boundary of a given normal domain D. *The first mixed boundary consists in finding a solution $u(x, t)$ in D of the equation* (1.1) *which is continuous in $\overline{D}$ and which assumes the values of ψ on the normal boundary of D.* By the maximum principle it follows that there exists at most one solution. With the aid of Theorems 1, 2 we now proceed to prove the existence of a solution, by extending the method of Poincaré in Potential Theory.

We first define a notion analogous to the notion of a super-harmonic function.

DEFINITION. A function $v(x, t)$ defined and piecewise continuous in D is called a *super-L function* if it satisfies the following properties:

(i) The discontinuity is (for every fixed x) of the first kind in t, the set of points of discontinuity lies on a finite number of hyperplanes $t=\tau$ and $v(x, \tau+0)\geqq v(x, \tau-0)$; we define $v(x, \tau)=v(x, \tau-0)$.

(ii) For every rectangular domain D^0 of D $([D^0]^-\subset D)$ the following

property holds: If u is the solution of (1.1) in D^0 with the boundary values v (on the normal boundary of D^0) constructed in §2 by formula (2.9) (see the last paragraph of §2), then $v \geqq u$ in D^0.

We shall prove several properties for super-L functions. For simplicity we confine ourselves to $n=1$; the extension to general n will be obvious. Let A be an arbitrary rectangle in D, defined by $x_0<x<x_1$, $t_0<t<t_1$. Constructing (by (2.9)) a solution $u(x, t)$ of (1.1) with the boundary values of a given super-L function v, we conclude, that v satisfies the inequality

$$\begin{aligned} v(\xi, \tau) \geqq & \int_{x_0}^{x_1} v(x, t_0) G^*(x, t_0; \xi, \tau) dx + \int_{t_0}^{\tau} \left[va \frac{\partial G^*}{\partial x} \right]_{x=x_0} dt \\ & - \int_{t_0}^{\tau} \left[va \frac{\partial G^*}{\partial x} \right]_{x=x_1} dt \end{aligned} \tag{4.1}$$

for all $(\xi, \tau) \in A$. We claim that the converse is also true:

LEMMA 8. *If $v(x, t)$ satisfies* (i) (*in the above definition*), *and if for each* $(\xi, \tau) \in D$, (4.1) *holds for all* (x_0, t_0), (x_1, t_0) *close to* (ξ, τ) *and such that* $t_0<\tau$, $x_0<\xi<x_1$, *then* $v(x, t)$ *is super-L.*

Proof. For simplicity we give the proof only in case v is continuous. The general case can be dealt with by the method used in proving Lemma 9′ below. We further remark that Lemma 8 is not used in proving Theorem 3 below.

Since for solutions u of (1.1) in D^0 (given by (2.9)), (4.1) holds with the equality sign, all we have to show is that the function $z=v-u$ for which (4.1) holds and which vanishes on the boundary of a subdomain D^0 of $D([D^0]^- \subset D)$, cannot take negative values in D^0.

Suppose that is false, then there exist points where z takes its negative minimum m. By a well known argument we can show that there exists a point $(\bar{\xi}, \bar{\tau})$ in D^0 such that $z(\bar{\xi}, \bar{\tau})=m$ and such that for some $x_0<\bar{\xi}$ or $x_1>\bar{\xi}$ and close to ξ $u>m$. Consequently, on using (4.1), we get

$$\begin{aligned} z(\bar{\xi}, \bar{\tau}) > & \int_{x_0}^{x_1} m G^*(x, t_0; \bar{\xi}, \bar{\tau}) dx + \int_{t_0}^{\bar{\tau}} m \left[a \frac{\partial G^*}{\partial x} \right]_{x=x_0} dt \\ & - \int_{t_0}^{\bar{\tau}} m \left[a \frac{\partial G^*}{\partial x} \right]_{x=x_1} dt. \end{aligned} \tag{4.2}$$

From Lemma 6 it follows that the right side of (4.2) with $\bar{\xi}$, $\bar{\tau}$ replaced by ξ, τ, is a solution of (1.1) in the rectangle $x_0<\bar{\xi}<x_1$, $t_0<t<\bar{\tau}$, and it assumes on the normal boundary of this rectangle the value m. Using the minimum principle we conclude that the right side of (4.2) is $\geqq m$. Hence, $z(\bar{\xi}, \bar{\tau})>m$, which is a contradiction.

LEMMA 9. *Let $v(x, t)$ be a super-L function and let D^0 be an $(n+1)$-dimensional rectangle, $[D^0]^- \subset D$. Define a function $w(x, t)$ in the following way:*

$$w(x, t) = \begin{cases} v_0(x, t) & \text{in } [D^0]^-, \\ v(x, t) & \text{in } D - D^0, \end{cases}$$

where v_0 is the solution of (1.1) in D^0 with the boundary values of v, which is constructed by (2.9). Then $w(x, t) \leqq v(x, t)$ and $w(x, t)$ is a super-L function.

For the proof of Theorem 3 we shall not need to make use of Lemma 9 and therefore we shall not give here its proof. The following lemma (which is slightly easier to prove) will be sufficient for our purposes.

LEMMA 9′. *Let v be a super-L function in D which is continuous in D and let $\{D_m\}$ be a sequence of $(n+1)$-dimensional rectangles, $[D_m]^- \subset \overline{D} - \partial D$. Define a sequence $\{v_m\}$ of functions, by induction, in the following way:*

$$v_m(x, t) = \begin{cases} v_{m-1}(x, t) & \text{if } (x, t) \in D - D_m, \\ u(x, t) & \text{if } (x, t) \in \overline{D}_m \end{cases}$$

and $v_0 = v$. Here, $u(x, t)$ is the solution of (1.1) in D_m with $u = v_{m-1}$ on the normal boundary of D_m (which is constructed by (2.9)). Then $\{v_m(x, t)\}$ is monotone decreasing.

Proof. The proof is by induction on m. The inductive assumption is that $v_{m-1}(x, t)$ is discontinuous on at most $m-1$ hyperplanes $t = c_k$ $(k = 1, \cdots, m-1)$, that $v_{m-1}(x, c_k+0) \geqq v_{m-1}(x, c_k)$, and that $v_{m-1}(x, t)$ satisfies (4.1) at each point $(\xi, \tau) \in D$ and for all x_0, x_1 close to ξ, $x_0 < \xi < x_1$, and all $t_0 < \tau$ and close to τ.

We now construct $u(x, t)$ by using (2.9). $u(x, t)$ is discontinuous on the intersection of D_m with $t = c_k$. We may assume that $c_1 < c_2 < \cdots < c_{m-1}$, and denote $B_0 = D_m \cap \{t < c_1\}$, $B_k = D_m \cap \{c_k < t < c_{k+1}\}$ $(1 \leqq k \leqq m-2)$, and $B_{m-1} = D_m \cap \{t > c_{m-1}\}$. We may assume that all the domains B_i are nonempty. From the proof of Lemma 8 (for continuous v) we conclude that $v_{m-1} \geqq u$ in B_0. Hence, in particular, $v_{m-1}(x, c_1+0) \geqq u(x, c_1)$. Now, since u was constructed in D_m by formula (2.9), (by the maximum principle it follows that) we can represent u in $D_m - B_0$ also by a formula analogous to (2.9) with the base R replaced by the base of $D_m - B_0$, that is the base of B_1. On this base $v_{m-1} \geqq u$ (by what we have already proved). On the lateral boundary of B_1 $v_{m-1} = u$. Let P approach a point $Q = (x, c_1)$ on the intersection of the basis of B_1 with any of its lateral faces. Using (4.1) for $v_{m-1}(P)$ and representing $u(P)$ in terms of (2.9), and taking in both cases the same rectangle A, which contains Q on the edge of its base, we easily conclude that $\liminf (v_{m-1}(P) - u(P)) \geqq 0$ as $P \to Q$. We now use the fact that $v_{m-1} - u$ satisfies (4.1) and apply the proof of Lemma 8 (for continuous v). We conclude that $v_{m-1} \geqq u$ in B_2 and, in particular, $v_{m-1}(x, c_2+0) \geqq u(x, c_2)$. Continuing in this way, we find that $v_{m-1} \geqq u$ in D_m and, therefore, $v_{m-1} \geqq v_m$ in D. Moreover, it is easy to show that v_m

satisfies (4.1) at each point (ξ, τ) of D. We finally note that v_m might have a discontinuity on $t=c_k$ $(k=1, \cdots, m-1)$ with upward jump, and it might have a discontinuity on one additional hyperplane $t=c$ (on which the upper face of D_m lies) with $v_m(x, c+0) \geqq v_m(x, c)$. This completes the proof.

We shall mention two more properties of super-L functions: If v_1 and v_2 are super-L functions, then $v=\min(v_1, v_2)$ is also a super-L function. If v is sufficiently smooth, then v is super-L if and only if $Lv \leqq 0$.

We can now extend the method of Poincaré [19, pp. 322–329] and construct a generalized solution for the first mixed boundary problem. (This solution satisfies (1.1) in the usual sense, but its behavior on the boundary still has to be studied.) Here we make use of Theorems 1, 2, and Lemma 9′. Note that the decomposition of a polynomial $p(x, t)$ into a difference of two super-L polynomials (the analogue of [19, p. 329]) can be given by $(p(x, t)+Ct) - Ct$, where $C=\max |Lp(x, t)|$, the maximum being taken over $\overline{D}$. Note also that the domains we use to accomplish the *méthode de balayage* are only $(n+1)$-dimensional rectangles, since only for such domains we know how to solve the first mixed boundary problem with discontinuous boundary values. We also use rectangles, one of whose faces lies on the upper boundary of D (on $t=T$).

Constructing the generalized solution $u(x, t)$, which will be continuous in D and on $t=T$ $((x, T) \in \overline{D}, (x, T) \notin C)$, it remains to show, under certain assumptions on C, that $u(x, t)$ assumes on the normal boundary the values of the given function $\psi(x, t)$. We first introduce the following notion:

DEFINITION. For any point Q of ∂D we define a *barrier* v_Q to be a function continuous in $\overline{D}$ and satisfying the following properties:

(i) $v_Q \geqq 0$ in $\overline{D}$, $v_Q=0$ only at Q;

(ii) $Lv_Q \leqq -1$ in D.

As in the harmonic case [19, pp. 326–328] the existence of a barrier at a point (x^0, t^0) is sufficient in order that any generalized solution will converge at that point to the corresponding boundary value (see also [28, pp. 422–423]). Furthermore, it is enough to construct a barrier in the small.

As a barrier for $(x^0, t^0)=(x^0, 0)$ we can take

$$v(x, t) = |x - x^0|^2 + Kt.$$

If K is sufficiently large, then $Lv \leqq -1$.

To construct barriers for $(x^0, t^0) \in C$, we have to impose some conditions on C. We shall first consider one example, analogous to Poincaré's criterion [19, p. 329].

DEFINITION. C is said to possess P-property at a point (x^0, t^0) on it, if there exists a sphere $|x-x'|^2+(t-t')^2=R_0^2$ passing through (x^0, t^0) and having (x^0, t^0) as its only common point with $\overline{D}$, and if $x' \neq x^0$.

We shall prove that if P-property is satisfied then there exists a barrier at the point (x^0, t^0). Take

$$v(x, t) = \frac{1}{R_0^{2\alpha}} - \frac{1}{R^{2\alpha}} \quad \text{where} \quad R^2 = |x - x'|^2 + (t - t')^2.$$

If α is large enough then, since we can take $|x-x'|$ bounded from below by a positive constant,

$$Lv = -\frac{4\alpha(\alpha+1)}{R^{2\alpha+4}} \sum_i a_i(x_i - x_i')(x_j - x_j') + \frac{2\alpha}{R^{2\alpha+2}} \sum_i a_i$$
$$+ \frac{2\alpha}{R^{2\alpha+2}} \sum_i b_i(x_i - x_i') + cv - \frac{2\alpha(t-t')}{R^{2\alpha+2}} \leqq -1.$$

Since also $v>0$ in $\overline{D} - \{(x^0, t^0)\}$ and $v(x^0, t^0) = 0$, v is a barrier in the small.

We now consider the case in which the condition $x' \neq x^0$ is not satisfied. We shall assume that the hyperplane $t = t^0$ is tangent to C at (x^0, t^0). In this case, different situations may arise. We shall consider a few typical cases.

(1) If there exists an $(n+1)$-dimensional rectangle contained in D and containing (x^0, t^0) in the interior of its upper base then, using repeatedly this subdomain in the process of the *balayage*, we get a solution which is continuous at (x^0, t^0) when $(x, t) \to (x^0, t^0)$ from within the rectangle.

(2) If the function $t - t^0 = \phi(x - x^0)$ of the boundary surface near (x^0, t^0) satisfies

$$1 + \left[\sum_i a_i(x, t) \frac{\partial^2 \phi(x - x^0)}{\partial x_i^2} \right]_{(x^0, t^0)} > 0 \tag{4.3}$$

and D is on the side $t - t^0 > \phi(x - x^0)$, then

$$K\{ |x - x^0|^4 + (t - t^0) - \phi(x - x^0)\}$$

is a barrier in the small for a sufficiently large constant K.

(3) In the case in which D lies below the curve $t - t^0 = \phi(x - x^0)$, barriers need not exist.

For $n = 1$, and for the heat equation, necessary and sufficient conditions for existence of barriers were given by Petrowski [27] (see also Gevrey [16, p. 323]). He proved that if the boundary curves, denoted by $x = \psi_i(t)$ $(i = 1, 2)(\psi_1(t) \leqq \psi_2(t))$, satisfy

$$\begin{aligned} \psi_1(t+h) - \psi_1(t) &\geqq -2(h \log \rho(h))^{1/2}, \\ \psi_2(t+h) - \psi_2(t) &\leqq 2(h \log \rho(h))^{1/2} \end{aligned} \tag{4.4}$$

for all $\epsilon_0 \leqq h < 0$ (for some fixed negative ϵ_0) where $\rho(h)$ is a monotone function which decreases to zero as $|h| \searrow 0$ and if

$$\int_{\epsilon_0}^{0} \frac{\rho(h) \, |\log \rho(h)|^{1/2}}{|h|} \, dh = \infty \tag{4.5}$$

then barriers exist at each point. If, however, the inequalities (4.4) are re-

versed at least for one t, and if $\rho(h)$ satisfies, instead of (4.5), the inequality

$$\int_{\epsilon_0}^{0} \frac{\rho(h)\,|\log \rho(h)|^{1/2}}{|h|}\,dh < \infty, \tag{4.6}$$

then at some points of C barriers do not exist.

Petrowski's results were extended by Fortet [13] to equations of the form $u_{xx}-u_t=b(x, t)u_x$.

Let us now note that the assumption $c(x, t)\leqq 0$ made throughout the discussion of the first mixed boundary problem can be abandoned since, for general $c(x, t)$, the substitution $u=e^{\alpha t}w$ ($\alpha=\max c(x, t)$) transforms the equation $Lu=0$ into the equation $(L-\alpha)w=0$ in which the coefficient of w is nonpositive. On the other hand, that substitution does not affect the other assumptions and assertions. We also note that all the previous results remain unchanged if we replace (B) (in §1) by the weaker assumption that the functions in (1.3) are Hölder continuous in closed subsets of D and the coefficients of L are continuous in $\overline{D}$.

The following theorem sums up our results concerning the first mixed boundary problem. For simplicity, it is not stated in the most general form.

THEOREM 3. *Let D be a normal domain and assume that L (given by* (1.1)) *is parabolic in $\overline{D}$, that the coefficients of L are continuous in $\overline{D}$ and that the functions* (1.3) *are Hölder continuous in closed subsets of D. Assume that at each point of C either P-property is satisfied, or the tangent hyperplane is of the form $t=$*const., *D is on the side $t-t^0>\phi(x-x^0)$ and* (4.3) *is satisfied. Then for any continuous boundary values, there exists a unique solution of the first mixed boundary problem.*

REMARK. Theorem 3 is true also for $T=\infty$.

We can now use Theorem 3 in order to accomplish the *méthode de balayage* by normal subdomains which possess A-property (mentioned in Theorem 1), and which tend to D. We then can prove (as in [19]) that the generalized solution is independent of both the particular choice of the sequence of subdomains, and the particular extension of the boundary values into a continuous function in $\overline{D}$.

Instead of following Poincaré's method one can also follow the Perron method and thus derive Theorem 3. An elegant treatment of the Perron method for second order elliptic equations is given in [25].

We now extend Theorem 3 to the case of nonhomogeneous equations.

THEOREM 4. *Let all the assumptions of Theorem* 3 *be satisfied and assume, in addition, that the coefficients of L are Hölder continuous in $\overline{D}$. Let $f(x, t)$ be Hölder continuous in $\overline{D}$ and let ψ be a continuous function of ∂D. Then there exists one and only one solution of the equation $Lu=f$ in D which coincides with ψ on ∂D.*

Proof. By [22] it follows that we can extend f and the coefficients of L into Hölder continuous functions in a cylinder D^* which contains D. Let $\Gamma(x, t; \xi, \tau)$ be the fundamental solution constructed by Pogorzelski [29] for general second order parabolic equations with Hölder continuous coefficients. By [29], the function

$$v(x, t) = \iint_{D^*} \Gamma(x, t; \xi, \tau) f(\xi, \tau) d\xi d\tau$$

is a particular solution of $Lv = f$ in D^*. Let w be a solution of $Lw = 0$ in D with the boundary values $\psi - v$ on D. (Its existence follows by Theorem 3.) Then $v + w$ is the desired solution.

REMARK. If D is a cylinder then it is not necessary to assume that f is Hölder continuous on the normal boundary ∂D; it is enough to assume only continuity on the boundary.

5. **Nonlinear equations in cylindrical domains.** We introduce the following notation: R is a bounded n-dimension domain with boundary ∂R; D_τ is the cylinder $\{(x, t); x \in R, 0 < t < \tau\}$; C_τ is the lateral boundary of D_τ, that is $\{(x, t); x \in \partial R, 0 \leqq t < \tau\}$; $\partial D_\tau = R + C_\tau$, and $\overline{D}_\tau$ is the closure of D_τ. We shall consider the following nonlinear system:

$$Lu = f(x, t) + k(x, t, u) \qquad \text{for } (x, t) \in D_\sigma, \tag{5.1}$$

$$u = \psi \qquad \text{for } (x, t) \in \partial D_\sigma \tag{5.2}$$

where σ is a fixed positive number. We shall need the following assumption:

(C) $k(x, t, u)$ is Hölder continuous in (x, t, u) when $(x, t) \in \overline{D}_\sigma$ and u varies in bounded sets, and

$$|k(x, t, u)| \leqq \lambda |u| + H |u|^\beta + H_0 \quad \text{for all} \quad (x, t) \in D_\sigma, \ -\infty < u < \infty, \tag{5.3}$$

where λ, H, H_0 are positive numbers and $0 \leqq \beta < 1$.

We shall also need to modify the assumptions (A), (B), as follows:

(A′) L is defined in $\overline{D}_\sigma$ and (1.2) holds for all (x, t) in $\overline{D}_\sigma$ and for all real vectors ξ, and g is positive.

(B′) The coefficients of L are Hölder continuous in $\overline{D}_\sigma$ and they are bounded by a positive constant M'. The functions in (1.3) are Hölder continuous in closed subsets of D_σ. Finally, $c(x, t) \leqq 0$ in D_σ.

We can now state the first result of this section.

THEOREM 5. *Assume that L satisfies* (A′), (B′), *that $f(x, t)$ is Hölder continuous in $\overline{D}_\sigma$, that $\psi(x, t)$ is a continuous function on ∂D_σ, that ∂R is of class C^2, and that $k(x, t, u)$ satisfies* (C) *with λ sufficiently small (depending only on g, M' and the diameter of R). Then the system* (5.1), (5.2) *has a solution.*

REMARK. If uniqueness is also proved (for instance, when $k(x, t, u)$ is monotone decreasing in u, or when $|k(x, t, u) - k(x, t, w)| \leqq \lambda_0 |u - w|$ where λ_0 is any constant (see [15])), then Theorem 5 easily yields a theorem of

existence and uniqueness of a solution in the infinite cylinder D_∞.

Proof. From the maximum principle [24] we can deduce that if u is a solution of (5.1), (5.2), then

$$\text{(5.4)} \qquad \underset{D_\sigma}{\text{l.u.b.}} \, |u| \leqq K\left[\underset{\partial D_\sigma}{\text{l.u.b.}} \, |\psi| + \underset{D_\sigma}{\text{l.u.b.}} \, |f| + \underset{D_\sigma}{\text{l.u.b.}} \, |k|\right],$$

where K is a positive constant depending only on g, M' and the diameter of R. Using (5.3) and assuming that $2\lambda K<1$ we easily get

$$\text{(5.5)} \qquad \underset{D_\sigma}{\text{l.u.b.}} \, |u| \leqq K_1$$

where K_1 is a constant independent of u. Hence, without loss of generality we may change the definition of $k(x, t, u)$ for $|u|>K_1$. We thus may assume that $k(x, t, u)$ satisfies the assumption (C) and, in addition,

$$\text{(5.6)} \qquad |k(x, t, u)| \leqq K_2 \qquad \text{for } (x, t) \in D_\sigma, \ -\infty < u < \infty$$

where K_2 is an appropriate constant.

We now introduce the Banach space X of functions $v(x, t)$ continuous in $\overline{D}_\sigma$ and with the uniform norm:

$$\|v\| = \underset{(x,t)\in D_\sigma}{\text{l.u.b.}} \, |v(x, t)| .$$

The closed convex set of functions v defined by $\|v\| \leqq N$ is denoted by X_N. For $v \in X_N$ we define a transformation $w=Tv$ as follows:

$$\text{(5.7)} \qquad w(x, t) = u_0(x, t) + \iint_{D_t} G(x, t; \xi, \tau) k(\xi, \tau, v(\xi, \tau)) d\xi d\tau.$$

Here u_0 is the solution of (5.1), (5.2) with $k\equiv 0$ whose existence follows from Theorem 4 (note that the assumption that ∂R is of class C^2 implies the P-property) and $G(x, t; \xi, \tau)$ is Green's function constructed by Pogorzelski [30] for general second order parabolic equations with Hölder continuous coefficients provided the boundary is of class $C^{1+\epsilon}$ for some $\epsilon>0$. (It is not proved in [30] that G is differentiable on the boundary.) The integral on the right side of (5.7), denoted by $z(x, t)$, satisfies the following properties (see [30]):

(i) As $(x, t)\to\partial D_\sigma$, $z(x, t)$ tends uniformly to zero.

(ii) If $\tilde{k}(\xi, \tau)\equiv k(\xi, \tau, v(\xi, \tau))$ is bounded uniformly with respect to (ξ, τ) and v, then $z(x, t)$ is Hölder continuous in $\overline{D}_\sigma$, with any fixed exponent $\mu<1/2$ and with a coefficient independent of v.

(iii) If $\tilde{k}(\xi, \tau)$ is Hölder continuous in closed subsets of D_σ then $\partial z/\partial t$, $\partial z/\partial x_i$, $\partial^2 z/\partial x_i \partial x_j$ are continuous functions in D_σ and $Lz=\tilde{k}(x, t)$.

We shall prove that T has a fixed point. Using (5.6) we conclude that the functions w are bounded independently of v in X. Hence, if N is sufficiently

large then T maps X_N into itself. By (ii) it follows that the integrals on the right side of (5.7), as v varies on X_N, are equi-continuous in $\overline{D}_\sigma$. Hence, the set TX_N is compact. Finally, since k is continuous in $(x, t)\in\overline{D}_\sigma$ and $\|v\|\leqq N$, we find that T is a continuous transformation of X_N. Applying Schauder's fixed point theorem [37] we conclude that T has a fixed point u, that is $u=Tu$.

Substituting $v=w=u$ in (5.7) we conclude (by (ii)) that $u(x, t)$ is Hölder continuous in closed subsets of D_σ. Hence, by (iii), u is a solution of (5.1). Using (i) we also find that u also satisfies (5.2).

The proof of Theorem 5 suggests another result. In order to state it, we shall need the following assumption.

(C′) $k(x, t, u)$ is Hölder continuous in (x, t, u) when $(x, t)\in\overline{D}_\sigma$ and u varies in bounded intervals.

THEOREM 6. *If all the assumptions of Theorem* 5 *are satisfied with the exception of* (C) *which is replaced by* (C′), *then the assertion of Theorem* 5 *holds provided* σ *is sufficiently small.*

Proof. Unlike the proof of Theorem 5, we now cannot change the definition of $k(x, t, u)$ so as to obtain (5.6). However we notice that

$$\iint_{D_t} |G(x, t; \xi, \tau)|\, d\xi d\tau \leqq H't \leqq H'\sigma \tag{5.8}$$

where H' depends only on the domain R and on the Hölder coefficients of the coefficients of L. Put

$$H'' = \underset{D_\sigma}{\text{l.u.b.}} |u_0(x, t)|\,. \tag{5.9}$$

Then, if we take $N=H''+1$ and choose σ such that

$$\left[\underset{(\xi,\tau)\in D_\sigma;\,|u|\leqq N}{\text{l.u.b.}} |k(\xi, \tau, u)|\right] H'\sigma \leqq 1 \tag{5.10}$$

then, by (5.8), (5.9), (5.10) and (5.7) we conclude that T maps X_N into itself. We can now complete the proof of the theorem almost word by word as in the proof of Theorem 5.

Generalization of Theorems 5, 6. If we use [14, Theorem 5] instead of Theorem 3 (in proving the existence of u_0 in (5.7)), then we derive the following corollary:

COROLLARY. *Theorems* 5, 6 *remain true with* L *being the operator*

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, t)\frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t}$$

which is assumed only to be parabolic in $\overline{D}_\sigma$ *and to have Hölder continuous coefficients in* $\overline{D}_\sigma$.

We remark that it is not necessary to assume the Hölder continuity of both $f(x, t)$ and $k(x, t, u)$ on the normal boundary ∂D. It is enough to assume continuity. We also remark that if the Hölder coefficients of L are bounded in D_∞ and if uniqueness holds (see the remark which follows Theorem 5) then Theorem 5, applied to an increasing sequence of domains yields existence and uniqueness in the whole cylinder D_∞. We finally remark that Theorem 5 remains true for *arbitrary* constant λ, but we omit here details of the proof.

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